

Variational formulation of sandpile dynamics

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Principle of stationary action was applied to derive a system of governing equations and boundary conditions describing dynamics of sandpile avalanches. The derived general system of equations for the sandpile dynamics is demonstrated to include equations of flow of granular material down a rigid wedge as a particular case. It is shown that the variational principle can be readily implemented in a numerical algorithm.

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I. INTRODUCTION

Statics and dynamics of granular materials were investigated quite intensively due to their importance in various naturally occurring phenomena (landslides, rockfalls, desert dunes evolution, sediment transport in rivers, etc.) and engineering applications (transportation and storage of coal, gravel, grain, etc.) [1–4]. An interesting type of granular flow is a surface flow, which occurs in a relatively thin boundary layer and does not penetrate into the bulk of the material [2]. Much attention was paid to investigation of dynamics of sandpiles which are built on a support surface by dropping of individual grains of granular material onto the free surface of the pile.

Consider a sandpile which is building on a flat support surface. Let grains of the material to be added one by one. The grains will accumulate on the pile surface until the surface slope reaches a critical angle of repose α_0 . After that, disturbances of the free surface due to grains added to the system are balanced, on average, by the granular material avalanches. The occurrence of the avalanches is spontaneous; the grains are able to accumulate on the pile surface and to pour suddenly down the slope. Thus sandpiles organize themselves into a statistically stable state.

The sandpile serves as a paradigm of the theory of self-organized criticality (SOC) proposed by Bak, Tang, and Wiesenfeld [5]. The prototype of SOC has been the sandpile cellular automata model. It was found that there were indeed no characteristic length or time scales, i.e., uncorrelated avalanches of all time and length scales are present. However, experiments did not fully confirm the prediction of the theory [6]. The sandpiles were subsequently perturbed by the addition of a single grain at the center of the pile. The next grain was dropped only after a full relaxation of avalanches caused by a previous grain. Pile mass fluctuations and a probability distribution of the granular material slides were measured for conical piles with different base diameters. The observed behavior of sufficiently small sandpiles was found to be in a good agreement with SOC theory. However, with increasing base diameter, the probability of the large avalanches increases while the probability of small ones decreases. Nearly all mass flow of a sandpile built on 3-in plate occurred

through large periodic avalanches.

Another important difference between the 3-in sandpile and smaller ones is the depth of the region where the flow occurs. Most of the grains which flow off the smaller sandpiles appear to originate near the sandpile surface, whereas the large avalanches observed for the 3-in pile result from the flow of sand further below the surface.

The ability of large masses of granular material to travel long distances indicates that those large avalanches are driven by strong inertial forces. A sandpile cellular automata model which takes into account these effects was proposed in [7]. The angle of repose was supposed to be a function of energy accumulated by the toppled particles. Due to this modification the model was able to reproduce the transition from the scale-invariant relaxation to oscillations.

The grains in experiments [6] were added to the system very slowly relative to the sandpiles' relaxation rate. It is intuitively clear that inertial effects only increase with the increasing of the intensity of the granular material inflow. The direct observations of stockpiles of sorted crushed stone suggest that real piles of granular material do demonstrate sudden collapselike inertia driven rearrangements [8]. Massive avalanches typically involve large blocs of the granular material and, therefore, the determination of a cause or location of the slide initiation is largely difficult. These avalanches suddenly accelerate down the slope and move through a long distance, incorporating additional particles into the motion. Large slides always decrease the inclination of the free surface of the sandpile.

Although the real processes of the grain redistribution is intermittent, the results of experiments on building sandpiles [9] allow us to suppose that some mean stable shape can be determined in the general case. A successful continual model of pile formation which provides quantitative results was proposed by Prigozhin [8,10]. In these studies the granular material was assumed to be characterized by one angle of repose α_0 . The inertial effects were neglected, i.e., the system was driven by gravitational and frictional forces only. Thus this model does not exhibit any collapselike behavior. The mathematical model of heap evolution was proved to be a dual formulation of a time-dependent quasivariational inequality.

In spite of its elegance and mathematical irreproachability, this method has a limited range of applications. The inequality used in [8,10] is a variational one only if the support

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surface of the pile is inclined at an angle less than the angle of repose everywhere. In the opposite case it becomes a quasivariational inequality which requires time consuming iterative treatment. In addition, this model was formulated in an axiomatic manner without any physical justification, and its possible generalizations are not clear.

Indeed, it can be showed that the obtained quasivariational inequality is equivalent to the requirement of the maximum dissipation rate. The latter principle was extensively used in the theory of plasticity, and it is known to be applicable only to quasistatic processes which proceed with an infinitely small rate. Catastrophiclike processes with strong inertial effects must be described by more general models which are based on the first principles of physics [11].

In this work the variational principle of stationary action was applied to a description of sandpile evolution. It is showed that a particular case of the model developed is equivalent to the model described in [8,10]. However, application of the principle of stationary action allows to take into account the inertial effects driving heap evolution, i.e., avalanches.

II. VARIATIONAL PRINCIPLE

We start with a generalized form of the principle of stationary action [11,12]:

$$\delta^* \int_{\Omega} \int_{t_1}^{t_2} (T-U) d\Omega dt + \int_{\Omega} \int_{t_1}^{t_2} (F_{\alpha} + \Lambda_{\alpha}) \delta^* a_{\alpha} d\Omega dt + \int_{\Gamma} \int_{t_1}^{t_2} (F_{\alpha} + \Lambda_{\alpha}) \delta^* a_{\alpha} d\Gamma dt = 0, \quad (1)$$

where density of Lagrangian $L = T - U$ is the difference between the kinetic and potential energies of the system, F_{α} are nonconservative body and surface generalized forces which cannot be represented as a gradient of a scalar potential (e.g., heat flux, viscous stress, etc.), Λ_{α} are generalized forces of reactions of constraints, and a_{α} are parameters of the system (e.g., configuration, entropy, etc.).

The symbol δ^* denotes total variation, while the symbol δ means variation at a fixed moment of time:

$$\delta^* a_{\alpha} \stackrel{\text{def}}{=} a'_{\alpha}(t', x') - a_{\alpha}(t, x) = \delta a_{\alpha} + \frac{da_{\alpha}}{dt} \delta \tau,$$

where $\delta \tau$ is "variation of time" $d/dt = \partial/\partial t + v_j \partial/\partial x_j$, $v_j = dX_j/dt$ is velocity, and X_j is displacement.

A parameter at a fixed point changes due to two reasons: its own variation and its transfer by virtual displacements [11],

$$\delta a_{\alpha} = \tilde{\delta} a_{\alpha} + \delta X_j \nabla_j a_{\alpha},$$

where $\tilde{\delta} a_{\alpha}$ is a local variation of the parameter a_{α} , and δX_j are virtual displacements. Therefore,

$$\delta^* a_{\alpha} = \tilde{\delta}^* a_{\alpha} + \delta^* X_j \nabla_j a_{\alpha}$$

$$\tilde{\delta}^* a_{\alpha} = \tilde{\delta} a_{\alpha} + \frac{\partial a_{\alpha}}{\partial t} \delta \tau.$$

Equation (1) for the virtual processes can be regarded as a generalization of the first law of thermodynamics. Indeed, this variational principle coincides with energy conservation law when variations δ are replaced by differentials d , i.e., when we take into account a real motion only.

III. PILE OF GRANULAR MATERIAL WITH COULOMB FRICTION

Generally the problem of sandpile dynamics is formulated as follows. Let a cohesionless granular material with bulk density ρ be poured down onto a rigid support surface with profile $h_0(x, y)$, where $(x, y) \in \Omega^*$ and forms a heap with a free boundary $h(t, x, y)$. All the material lies above the support surface, i.e.,

$$h \geq h_0. \quad (2)$$

The surface density of the material which falls onto the area $d\Omega$ during time interval dt is $\rho w(t, x, y)$. Denote the mass flux in the horizontal plane by $\rho \vec{q}$. Assume that the part Γ_1^* of the boundary of the domain Ω is bounded by impermeable wall, i.e.,

$$q_n|_{\Gamma_1} = 0, \quad (3)$$

On the remaining part Γ_2^* of the boundary the material may leave the system freely, and

$$h|_{\Gamma_2} = h_0.$$

The problem is to determine the time dependence of the height of a granular pile $h(t, x, y)$ and vector of horizontal mass flux $\rho \vec{q}$.

In order to demonstrate that the governing equations for the sandpile evolution may be determined with the aid of the principle of a stationary action, consider the simplest case of a pile of granular material with Coulomb friction, and neglect inertial effects in the flowing granular material. In this case the surface density of Lagrangian per unit area of domain Ω equals the potential energy with a minus sign:

$$L = -\rho g \int_0^h x dx = -\frac{\rho g}{2} h^2,$$

where density ρ is supposed to be constant, and g is an acceleration of gravity.

There are three contributions into the nonpotential part of the variational equation (1): dissipation due to friction in the granular material, virtual influx of the Lagrangian through the boundary of the domain, and potential energy which is acquired by grains falling during the time interval $\delta \tau$.

Determining the friction force \vec{F} requires a detailed description of the friction mechanisms in a granular material. Such mechanisms were discussed in [13,14], and it was confirmed that forces acting on a layer of sand sliding down a rough wedge corresponds to the Coulomb-like friction, i.e., $\vec{F} \propto \vec{u}/|u|$, where \vec{u} is velocity of granular material.

On the other hand, it is known that sandpile avalanches occur only if the pile surface is inclined at the critical angle with respect to the horizontal plane. Thus an interesting simi-

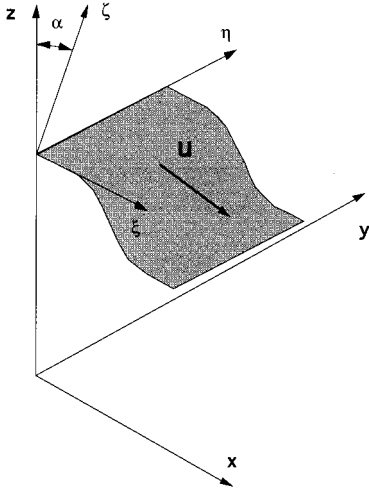


FIG. 1. Schematic view of the surface.

larity exists between plasticity theory and granular pile evolution. According to the ideal plasticity model of von Mises, the virtual dissipation in a plastic body is given by the following equation [15]:

$$\delta E = \sigma_{ij} \delta S_{ij} = -k \frac{s_{ij}}{\sqrt{s_{ij}s_{ij}}} \delta S_{ij},$$

where σ_{ij} is a deviator of a stress tensor, s_{ij} is a tensor-deviator of the deformation rate, δS_{ij} is a tensor-deviator of the virtual deformation, and k is a threshold shearing stress. Using this similarity the mass flux density of the granular material can be viewed as a deformation rate, while the tangent of angle of repose γ can be viewed as a threshold shearing stress k . Thus we postulate the dissipation per unit area of domain Ω to be in the following Mises-like form:

$$\delta A = \rho \vec{F} \cdot \delta \vec{Q} = -\gamma \rho g \frac{\vec{q}}{|\vec{q}|} \cdot \delta \vec{Q}$$

where δA is energy which is dissipated during displacement of the granular material with mass m at a distance $\delta \vec{r}$ such that $m \delta \vec{r} = (\rho d\Omega) \delta \vec{Q}$, $\vec{q} = (\partial/\partial t) \vec{Q}$.

It is important to note that although \vec{F} is a two-dimensional vector, it does not equal to the horizontal component of the friction force which acts on the flowing granular layer, but is a vector which being multiplied by the material flow rate \vec{q} equals the dissipation rate per unit area of domain Ω . Consider a motion of a portion of granular material with mass m and velocity $\vec{u} = (u_1, u_2, u_3)$ on a support surface with height h . Introduce the local coordinates (ξ, η, ζ) at the surface (Fig. 1). In these coordinates velocity of granular material $\vec{u} = (u_\xi, u_\eta, 0)$. The expression for Coulomb-like friction force $\vec{F}^c = (F_\xi^c, F_\eta^c, 0)$ is given in [16],

$$F_\xi^c = \gamma m g \cos \alpha \frac{u_\xi}{|u|},$$

$$F_\eta^c = \gamma m g \cos \alpha \frac{u_\eta}{|u|},$$

where γ is coefficient of friction, α is angle of inclination of the surface, and $\cos \alpha$ can be expressed via $\vec{\nabla} h$ as follows:

$$\cos \alpha = [1 + (\vec{\nabla} h)^2]^{-1/2}$$

Consider the expression for the dissipation rate,

$$\varepsilon = F_\xi^c u_\xi + F_\eta^c u_\eta.$$

It is clear for simple geometrical reasons that

$$u_\xi = \left[1 + \left(\frac{\partial h}{\partial x_i} \frac{u_i}{|u|} \right)^2 \right]^{1/2} u_1, \quad i=1,2,$$

$$u_\eta = \left[1 + \left(\frac{\partial h}{\partial x_i} \frac{u_i}{|u|} \right)^2 \right]^{1/2} u_2, \quad i=1,2.$$

Therefore the formula for dissipation rate can be written in the following form:

$$\varepsilon = m g \gamma [1 + (\nabla h)^2]^{1/2} \left[1 + \left(\frac{\partial h}{\partial x_i} \frac{u_i}{|u|} \right)^2 \right]^{1/2} \frac{u_i^2}{|u|}, \quad i=1,2.$$

Thus in a case of Coulomb-like friction, we can use the expression

$$m F_i = m g \gamma [1 + (\nabla h)^2]^{-1/2} \left[1 + \left(\frac{\partial h}{\partial x_i} \frac{u_i}{|u|} \right)^2 \right]^{1/2} \frac{u_i}{|u|} \quad (4)$$

as a definition of dissipative force \vec{F} . Notably, a coefficient near $m g \gamma (u_i/|u|)$ in Eq. (4) determines the transformation of force from the local coordinates to (x, y, z) coordinates in a general case for arbitrary friction law.

When the material flow velocity is parallel to the support surface profile gradient $(u_1, u_2) \propto (\partial h/\partial x_1, \partial h/\partial x_2)$, the above formula for dissipative force \vec{F} can be simplified:

$$m F_i = m g \gamma \frac{u_i}{|u|},$$

i.e., the assumption of Mises-like friction is equivalent to the assumption that the material flow velocity is directed toward the steepest descent of the support surface profile. This assumption will be justified below.

The virtual flux of the Lagrangian through the boundary equals the potential energy which is acquired by the virtual mass flow rate of grains:

$$\vec{\Phi}_L = -\rho g h \vec{\delta}^* \vec{Q},$$

The potential energy which is acquired by grains falling during the time interval $\delta \tau$ equals $\rho g w h \delta \tau$. Therefore the variational equation (1) for a granular material with Coulomb friction and neglecting inertial forces reads

$$\begin{aligned} & \rho g \int_{t_1}^{t_2} \int_{\Omega} \left\{ h \bar{\delta}^* h + \gamma \frac{\vec{q}}{|q|} \cdot \bar{\delta}^* \vec{Q} - w h \delta \tau \right\} d\Omega dt \\ & + \rho g \int_{t_1}^{t_2} \oint_{\Gamma} h \bar{\delta}^* \vec{Q} \cdot d\vec{n} dt = 0. \end{aligned} \quad (5)$$

Note that variations in Eq. (5) are not independent. Obviously, the variational principle is invariant if h is replaced by $h + \tilde{h}$, where \tilde{h} is an arbitrary constant. Substituting $h + \tilde{h}$ into Eq. (5) and demanding that the coefficient near \tilde{h} vanish, we obtain

$$\bar{\delta}^* h = -\vec{\nabla} \cdot \bar{\delta}^* \vec{Q} + w \delta \tau. \quad (6)$$

Setting $\delta \tau = 0$ in (6) we recover the continuity equation for the synchronous variations of the governing parameters:

$$\bar{\delta} h = -\vec{\nabla} \cdot \bar{\delta} \vec{Q}. \quad (7)$$

From Eqs. (6) and (7) we obtain the continuity equation for the real process:

$$\frac{\partial h}{\partial t} = -\vec{\nabla} \cdot \vec{q} + w. \quad (8)$$

Substituting (7) into (5), setting $\delta \tau = 0$ and using Gauss' formula, we obtain the following equation:

$$\vec{\nabla} h = -\gamma \frac{\vec{q}}{|q|}. \quad (9)$$

The material flow is directed toward the steepest descent, therefore formula (9) confirms the assumption of Mises-like dissipation. Note that Eq. (9) is valid only if $q \neq 0$ and $h \geq h_0$. Thus Eq. (9) contains the implicit assumptions that the granular material is actually flowing, and that the moving layer is located above the support surface.

Equation (9) implies that

$$|\vec{\nabla} h| = \gamma$$

for the flowing material, i.e., we have obtained the model with one angle of repose.

The governing equations that we derived were known before. However, it is important to develop a variational approach for the simplest case before considering more complex situations. Although Eq. (5) does not demonstrate the full potential of the variational approach, it can be used for constructing a numerical algorithm (see the Appendix).

Notably Eq. (5), unlike the equation used in [8,10], is a purely variational equation for arbitrary inclination of the support surface.

IV. AVALANCHES

Use of the Coulomb friction law leads to the model described by Eqs. (5) and (6), which does not exhibit any collapse-like avalanches. In order to incorporate larger avalanches into this model, one must take into account that a real friction in the granular material is not monotone. When an initial portion of sand starts to slide down the heap, shear-

ing induced in the moving layer leads to dilatation and therefore reduces friction. The latter results in particle acceleration and in a transfer of a "superfluous" momentum to lower granular layers, involving them in the motion. In order to describe such avalanches it is necessary to take the effects of inertia of granular material into account.

Consider the granular material avalanche which propagates down the pile surface with height h . The flow occurs in a layer with thickness l , and the velocity of the center of gravity of the moving layer is $\vec{u} = (u_1, u_2, u_3)$. At the closed part of the boundary Γ_1 the normal component of the velocity is zero, and on the open part the depth of the stationary bulk of granular material is zero. Thus the boundary conditions read

$$u_i n_i |_{\Gamma_1} = 0, \quad (10)$$

$$h |_{\Gamma_2} = h_0,$$

where n_i is the unit vector normal to the boundary. Hereafter all indices vary from 1 to 2 (e.g., $i = 1$ and 2), since we consider only the horizontal flow component.

The thickness of the flowing layer is positive, and no material lies below the support surface. Thus the system is under the following constraints:

$$-l \leq 0, \quad (11)$$

$$h_0 - h \leq 0. \quad (12)$$

In order to formulate equations for the sandpile avalanches, we must take into account kinetic and potential energies as well as dissipative forces in the flowing granular material.

The formula for the surface density of kinetic energy of the moving layer reads

$$\rho l T = \frac{\rho l}{2} \sum_{\alpha=1}^3 u_{\alpha}^2. \quad (13)$$

The velocity component in the vertical direction u_3 can be approximated by an arithmetical mean of the upper and lower boundaries of the sliding granular layer, $u_3 = \frac{1}{2} (u_3^{\text{upper}} + u_3^{\text{lower}})$, which satisfy the following kinematic equations:

$$\begin{aligned} \vec{u}_3^{\text{lower}} &= u_i \frac{\partial h}{\partial x_i}, \\ \vec{u}_3^{\text{upper}} &= \frac{\partial(h+l)}{\partial t} - w + u_i \frac{\partial(h+l)}{\partial x_i}. \end{aligned}$$

The formula for the surface density of the potential energy reads

$$U = \frac{\rho g}{2} (h+l)^2, \quad (14)$$

It is assumed that shearing does not influence the layer density [13].

Virtual flux of the Lagrangian through the boundary of the domain Ω is given by the following formula:

$$\vec{\Phi}_L = -\rho l \left[T - g \left(h + \frac{l}{2} \right) \right] \delta^* \vec{x}, \tag{15}$$

where $\delta^* \vec{X} = (\delta^* X_1, \delta^* X_2)$ is a virtual displacement. In formula (15) for Lagrangian flux, we take into account only the potential energy of the flowing granular layer. Assuming a particular form of the dissipative force $\rho \vec{F} = \rho(F_1, F_2)$ closes the model. The latter expressions were derived under the assumption that all parameters of the moving granular layer may be averaged over the layer's depth as it was done in [13].

Substituting Eqs. (13)–(15) into variational Eq. (1), multiplying the constraint (11) by Lagrangian multiplier $\rho \lambda$ and the constraint (12) by Lagrangian multiplier $\rho \mu$, we obtain the following variational equation:

$$\begin{aligned} & \rho \int_{t_1}^{t_2} \int_{\Omega} \{ l \tilde{\delta}^* T + [T - \lambda - g(h+l)] \tilde{\delta}^* l - [g(h+l) - \mu] \tilde{\delta}^* h \\ & - l F_i \delta^* X_i - g(h+l) w \delta \tau \} dt d\Omega \\ & + \rho \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ l \left[T - \rho g \left(h + \frac{l}{2} \right) \right] \delta^* X_i \right\} dn_i dt \end{aligned}$$

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega} \left\{ - \frac{d}{dt} \left[l u_i + l u_3 \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{l^2}{2} u_3 \right) \right] + l u_3 \left[\frac{\partial^2 h}{\partial x_i \partial x_j} u_j - \frac{\partial}{\partial x_i} \left(\frac{l}{2} \right) \frac{\partial u_j}{\partial x_j} \right] - \frac{\partial}{\partial x_i} \left(\frac{l^2}{2} \frac{\partial u_j}{\partial x_j} u_3 \right) - g l \frac{\partial(h+l)}{\partial x_i} - l F_i \right. \\ & \left. - l \frac{\partial \lambda}{\partial x_i} \right\} \delta X_i dt d\Omega + \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ \frac{d}{dt} \left(\frac{l^2}{2} u_3 \right) + \frac{l^2}{2} \frac{\partial u_j}{\partial x_j} u_3 - g \frac{l^2}{2} \right\} \delta X_i dn_i dt = 0 \end{aligned} \tag{20}$$

Setting in (20) the variations δX_i at the boundary equal to zero, we obtain the equations of motion

$$\begin{aligned} & \frac{d}{dt} \left[l u_i + l u_3 \frac{\partial h}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{l^2}{2} u_3 \right) \right] \\ & = l u_3 \left[\frac{\partial^2 h}{\partial x_i \partial x_j} u_j - \frac{\partial}{\partial x_i} \left(\frac{l}{2} \right) \frac{\partial u_j}{\partial x_j} \right] - \frac{\partial}{\partial x_i} \left(\frac{l^2}{2} \frac{\partial u_j}{\partial x_j} u_3 \right) \\ & - g l \frac{\partial(h+l)}{\partial x_i} - l F_i - l \frac{\partial \lambda}{\partial x_i}. \end{aligned} \tag{21}$$

Since variations δX_i are arbitrary, Eq. (20) yields the following boundary condition:

$$\left\{ \frac{d}{dt} \left(\frac{l^2}{2} u_3 \right) + \frac{l^2}{2} \frac{\partial u_j}{\partial x_j} u_3 - g \frac{l^2}{2} \right\} \Big|_{\Gamma_2} = 0. \tag{22}$$

Keeping the terms with variations of l and h in (16) and setting variations $\delta X_i = 0$ leads to the following equation which determines the depth of a moving layer:

$$\frac{1}{2} \sum_{\alpha=1}^3 u_{\alpha}^2 + \frac{\partial(l u_3 u_i)}{\partial x_i} - \frac{l}{2} \frac{\partial u_i}{\partial x_i} u_3 + \lambda - \mu = 0. \tag{23}$$

$$= 0. \tag{16}$$

As was noted before, the variations in the Eq. (16) are not independent. Replacing h by $h + h + \tilde{h}$ and repeating the analysis from Sec. III, we obtain the following continuity conditions:

$$\tilde{\delta} h + \tilde{\delta} l = - \frac{\partial(l \delta X_i)}{\partial x_i}, \tag{17}$$

$$\frac{\partial h}{\partial t} + \frac{\partial l}{\partial t} = - \frac{\partial(l u_i)}{\partial x_i} + w. \tag{18}$$

Using the equation of continuity (18) we can rewrite the expression for the vertical component of the velocity in the following form:

$$u_3 = u_i \frac{\partial h}{\partial x_i} - \frac{l}{2} \frac{\partial u_i}{\partial x_i}. \tag{19}$$

Setting $\tilde{\delta}^* h$ and $\delta \tau$ in (16) equal to zero, taking into account Eq. (17) and integrating by parts with respect to time and the spatial coordinates, we arrive at the following variational equation:

The derived system of partial differential equations (18), (21), and (23), and inequalities (11) and (12) for h , \vec{u} , l , λ , and μ , must be solved with the boundary conditions (12) and (22).

Since the depth of the moving layer l was assumed to be small compared with the characteristic length and height of the pile, it was shown in [13,16] that terms of order l^2 may be omitted in the equation of momentum conservation, but in variational equation (16) we kept them in order to derive the boundary condition (22). Neglecting these terms, we can rewrite Eqs. (21) and (19) in the following simplified form:

$$\frac{d}{dt} l \left(u_i + u_3 \frac{\partial h}{\partial x_i} \right) = l u_3 \frac{\partial^2 h}{\partial x_i \partial x_j} u_j - g l \frac{\partial(h+l)}{\partial x_i} - l F_i - l \frac{\partial \lambda}{\partial x_i}, \tag{24}$$

$$u_3 = u_j \frac{\partial h}{\partial x_j}. \tag{25}$$

The variational principle in the form (16) may be also used for obtaining the governing equations in the case when granular material is sliding down a rigid wedge. Differential equations for this case were obtained in [13,16] by averaging full equations of conservation mass and momentum over the moving layer depth.

Indeed, replacing h by h_0 and taking $\delta^*h=0$, we can rewrite the variational equation (16) in the following form:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\Omega} \{l \bar{\delta}^* T + [T - g(h_0 + l)] \bar{\delta}^* l - l F_i \delta^* X_i \\ & - g(h_0 + l) w \delta \tau\} dt d\Omega \\ & + \int_{t_1}^{t_2} \oint_{\Gamma} \left\{ l \left[T - \rho g \left(h_0 + \frac{l}{2} \right) \right] \delta^* X_i \right\} dn_i dt = 0. \end{aligned} \quad (26)$$

Repeating the above analysis we obtain the continuity equations

$$\frac{\partial l}{\partial t} = - \frac{\partial(l u_i)}{\partial x_i} + w, \quad (27)$$

$$\bar{\delta} l = - \frac{\partial(l \delta x_i)}{\partial x_i}.$$

Substituting the latter equations into the variational equality (26) and integrating by parts with respect to time and spatial coordinates, we obtain the equations of momentum conservation

$$\frac{d}{dt} l \left(u_i + u_3 \frac{\partial h_0}{\partial x_i} \right) = l u_3 \frac{\partial^2 h_0}{\partial x_i \partial x_j} u_j - g l \frac{\partial(h_0 + l)}{\partial x_i} - l F_i, \quad (28)$$

and the boundary condition

$$\left\{ \frac{d}{dt} \left(\frac{l^2}{2} u_3 \right) + \frac{l^2}{2} \frac{\partial u_j}{\partial x_j} u_3 - g \frac{l^2}{2} \right\} \Big|_{\Gamma_2} = 0.$$

In Eq. (28), small terms of order l^2 were neglected. Equations (27) and (28) in (ξ, η) coordinates coincide after inverse transformation of dissipative force (4) with equations which were derived in [13] and [16].

The specific form of the friction force is not discussed here. Friction in flowing sand was investigated by many authors (see, e.g., [13,14]), and it was confirmed that the main contribution into a friction force constitute Coulomb-like friction with weak shear rate dependence.

V. SEGREGATION IN PILE OF GRANULAR MATERIAL

Let us consider the evolution of a sandpile composed of a binary mixture of particles of the same density and with diameters d_1 and d_2 . If $0.39 \leq d_1/d_2 \leq 2.55$, the bulk density ρ does not depend upon the mixture composition, i.e., changes in the bulk density less than 5% [17]. Therefore, hereafter ρ is supposed to be constant.

Mass of the material of the i th type which falls onto the area $d\Omega$ during time interval dt is $\rho w_i(t, x, y) d\Omega dt$. Denote the mass flux of the i th component in the horizontal plane by $\rho \vec{q}_i$. Let $C_i(t, x, y, z), x, y \in \Omega, z \in [h_0, h - 0]$ be the mass fraction of the i th component of the granular material inside the stationary granular heap, i.e.,

$$C_i \geq 0, \quad C_1 = C, \quad C_2 = (1 - C).$$

Assume that the boundary Γ of the domain Ω is bounded by an impermeable wall, i.e.,

$$q_n|_{\Gamma} = 0.$$

The equations of material balance for each component of the granular material can be written as follows:

$$C \frac{\partial h}{\partial t} = - \vec{\nabla} \cdot \vec{q}_1 + w_1,$$

$$(1 - C) \frac{\partial h}{\partial t} = - \vec{\nabla} \cdot \vec{q}_2 + w_2.$$

Let us consider the main contributions to the rate of energy dissipation during particle slides down the heap. It is suggested [17] that the dissipation arises from kinetic energy lost in inelastic interparticle collisions, and from potential energy loss due to rubbing. Consider small discharge rates, so that inertial effects in the flowing granular material can be neglected, and take into account only the Coulomb-like friction during the particles sliding down the sandpile slope. Then the dissipation rate of energy of the mixture per unit area of domain Ω can be represented as a sum of dissipations rates of each component:

$$\delta A = - \rho g \left(\gamma_1 \frac{\vec{q}_1}{|q_1|} \cdot \vec{\delta} \vec{Q}_1 + \gamma_2 \frac{\vec{q}_2}{|q_2|} \cdot \vec{\delta} \vec{Q}_2 \right), \quad q_i = \partial Q_i / \partial t,$$

where $\gamma_i(C[x, y, h(x, y) - 0])$ depend on a composition of the upper layer of the stationary granular pile.

Keeping only synchronous variations of the governing parameters, we can rewrite the variational equation (5) in the following form:

$$\begin{aligned} & \rho g \int_{t_1}^{t_2} \int_{\Omega} \left\{ h \delta h + \gamma_1 \frac{\vec{q}_1}{|q_1|} \cdot \vec{\delta} \vec{Q}_1 + \gamma_2 \frac{\vec{q}_2}{|q_2|} \cdot \vec{\delta} \vec{Q}_2 - w_1 h \delta \tau \right. \\ & \left. - w_2 h \delta \tau \right\} d\Omega dt = 0, \end{aligned} \quad (29)$$

where

$$\delta h = - \vec{\nabla} \cdot \vec{\delta} \vec{Q}_1 - \vec{\nabla} \cdot \vec{\delta} \vec{Q}_2. \quad (30)$$

If the upper layer of the granular heap slides down, the composition of the granular material which starts to slide down the heap is determined by the composition of the adjacent granular layer which is involved in the motion:

$$\frac{C}{1 - C} = \frac{\vec{\nabla} \cdot \vec{\delta} \vec{Q}_1}{\vec{\nabla} \cdot \vec{\delta} \vec{Q}_2} \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \geq w_1, \quad (31)$$

where C is determined at the point $(x, y, [h(x, y) - 0])$.

When any component of the mixture is sedimented on the granular pile, the other component cannot leave the upper layer of the granular heap. Therefore,

$$\vec{\nabla} \cdot \vec{\delta} \vec{Q}_2 \leq 0 \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \leq w_1. \quad (32)$$

Multiplying constraint (31) by Lagrangian multiplier $\rho\chi_1$, and constraint (32) by Lagrangian multiplier $\rho\chi_2$, substituting this into the variational Eq. (29), using (30) and the Gauss formula, we obtain the following equations:

$$\nabla[h + \chi_1(1 - C)] = -\gamma_1 \frac{\vec{q}_1}{|q_1|} \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \geq w_1, \quad (33)$$

$$\nabla(h - \chi_1 C) = -\gamma_2 \frac{\vec{q}_2}{|q_2|} \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \geq w_1,$$

$$\nabla h = -\gamma_1 \frac{\vec{q}_1}{|q_1|} \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \leq w_1, \quad (34)$$

$$\nabla(h + \chi_2) = -\gamma_2 \frac{\vec{q}_2}{|q_2|} \quad \text{for } \vec{\nabla} \cdot \vec{q}_1 \leq w_1.$$

The above equations together with equations of material balance provide a closed description of the pile evolution.

Equation (34) shows that $\chi_2 \neq 0$ for $\gamma_1 \neq \gamma_2$. Since χ_2 is a Lagrangian multiplier, $\chi_2 \nabla \cdot \delta \vec{Q}_2 = 0$, and the second component does into sediment during sedimentation of the first component of the granular mixture. Therefore, the combination of the assumption of Coulomb friction and the small velocity of the slide yield the model which predicts the complete separation of components of granular material during avalanches. Such a separation can occur when intermittent avalanches are obtained when rotating a horizontal drum filled with a mixture of two granular materials slowly and continuously. Indeed, particle size segregation in a rotating drum was observed experimentally in [17].

VI. CONCLUSIONS

A variational form of the basic thermodynamics principles was applied to describe dynamics of sandpiles. A system of governing the equations and boundary conditions was derived. It is shown that the variational principle can be readily implemented in a numerical algorithm. Equations describing the flow of granular material down a rigid edge were demonstrated to be a particular case of the derived general system of equations for the avalanches dynamics in sandpiles. The derived system of equations of sandpile dynamics predicts the complete separation of a mixture of two granular materials during avalanches [18].

APPENDIX: NUMERICAL ALGORITHM

The variational method not only provides an analytical description of the problem, it can also be used for construct-

ing a numerical procedure. Consider the simplest case of a pile with Coulomb friction.

After discretizations in time, Eq. (8) and the formula for $\delta \vec{Q}$ at time k_2 read

$$h_k = h_{k-1} + (w - \vec{\nabla} \cdot \vec{q}_k) \Delta t,$$

$$\delta \vec{Q}_k = \delta \vec{q}_k \Delta t.$$

Then, substituting the above formulas into variational Eq. (5), after simple algebra we obtain the following minimization problem at time $k\Delta$:

$$\int_{\Omega} \left\{ \frac{\Delta t}{2} (\vec{\Delta} \cdot \vec{q}_k)^2 + \gamma |\vec{q}_k| - (h_{k-1} + w \Delta t) \vec{\nabla} \cdot \vec{q}_k \right\} d\Omega + \oint_{\Gamma_2} h_0 \vec{q}_k \cdot d\vec{n} \rightarrow \min. \quad (A1)$$

Constraint (2) and boundary condition (3) can be written as follows:

$$f = -h_0 - (h_{i-1} + w \Delta t) + \Delta t \vec{\nabla} \cdot \vec{q}_i \leq 0, \quad (A2)$$

$$q_n|_{\Gamma_1} = 0.$$

Equations (A1) and (A2) together with Eq. (8) provide a closed description of the system. This problem can be solved by duality method [19]. The simplest one leads to the following sequence of minimization problems:

$$\int_{\Omega} \left\{ \frac{\Delta t}{2} (\nabla \cdot \vec{q}_k^{(n)})^2 + \gamma \vec{p}_k^{(n)} \vec{q}_k^{(n)} - (h_{k-1} + w \Delta t) \nabla \cdot \vec{q}_k^{(n)} + r(f^{(n)} - \zeta^{(n)})^2 + \lambda^{(n)}(f^{(n)} - \zeta^{(n)}) \right\} d\Omega + \oint_{\Gamma_2} h_0 \vec{q}_k^{(n)} \cdot d\vec{n} \rightarrow \min,$$

$$\lambda^{(n)} = \sup\{[\lambda^{(n-1)} + \theta_1 r f(\vec{q}_k^{(n-1)})], 0\}, \quad 0 < \theta_1 < 2,$$

$$\vec{p}^{(n)} = \left(\vec{p}^{(n-1)} + \frac{\theta_2}{\gamma} \vec{q}_k^{(n-1)} \right) / \sup\left[1, \left(\vec{p}^{(n-1)} + \frac{\theta_2}{\gamma} \vec{q}_k^{(n-1)} \right) \right], \quad 0 < \theta_2 < 2,$$

$$\zeta^{(n)} = \sup\left[\left(f^{(n-1)} - \frac{\lambda^{(n)}}{2r} \right), 0 \right],$$

where \vec{p} and λ are Lagrangian multipliers, ζ is a slack variable, and r is a penalty parameter.

This problem of quadratic programming can be solved by relaxation methods or by reduction to linear equations of an elliptic type. The above algorithm converges for an arbitrary value of a penalty parameter r [19].

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